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A SPECTRAL IDENTITY BETWEEN SYMMETRIC SPACES

BY

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ABSTRACT

Let π be a cuspidal automorphic representation of GL_{2n}. We prove an identity between two spectral distributions on Sp_{2n} and GL_{2n} respectively. The first is the spherical distribution with respect to $Sp_n \times Sp_n$ of the residual Eisenstein series induced from π. The second is the weighted spherical distribution of π with respect to $GL_n \times GL_n$ and a certain degenerate Eisenstein series. A similar identity relates the pair (U_{2n}, Sp_n) and $(GL_n/E, GL_n/F)$ where E/F is the quadratic extension defining the quasi-split unitary group U_{2n}. We also have a Whittaker version of these trace identities.

1. Introduction

In this paper we apply the analysis of [JLR03] to additional cases and prove a spectral identity between representations on two different groups which are related by functoriality. This identity is the cuspidal part of a trace formula identity of the type that is traditionally obtained by geometric comparison. However, our

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method avoids the use of the trace formula and the identity is obtained more directly.

The input and the motivation for this identity and the trace formula stem from two sources. The first is the notion of a cuspidal representation π on a group G being **distinguished** by a subgroup H. This means that the **period** linear form

$$\ell_H(\varphi) = \int_{H(F)\backslash H(\mathbb{A})^{(1)}} \varphi(h)\, dh$$

is not identically 0 on π where $H(\mathbb{A})^{(1)}$ is a certain normal subgroup of $H(\mathbb{A})$ which is compact modulo $Z_G(\mathbb{A}) \cap H(\mathbb{A})$. Usually, H is a "large" subgroup of G — typically the fixed point subgroup of an involution of G. Both G and the involution are defined over a number field F and $\mathbb{A}$ is the ring of adèles of F. We refer the reader to [Jac97], [Jac] and [LR03] for a general discussion about periods and distinguished representations and their relation to functoriality, L-functions, arithmetic and geometry.

The second source is integral presentations of L-functions by Rankin–Selberg type integrals.

To link these two, consider the case where the part of the spectrum which is distinguished by H is residual. Let M be a Levi subgroup of a maximal parabolic subgroup G and let $E(\varphi, g, s)$ be an Eisenstein series induced from a cuspidal automorphic representation π of $M(\mathbb{A})$. Often, the L-function $L(s, \pi, \rho)$ (or product of these) appearing in the constant term of E has an integral presentation of the form $\int_{M_H(F)\backslash M_H(\mathbb{A})^{(1)}} \varphi(m)\mathfrak{E}(m,s)\, dm$ where M_H is a "large" subgroup of M, $\mathfrak{E}(\cdot, s)$ is a certain degenerate Eisenstein series on M and φ is in the space of π. In particular, the residue of that L-function at a special point s_0 is related to the period over M_H. Thus, the existence of a pole for the L-function, or equivalently, for the Eisenstein series, is characterized by π being distinguished by M_H. Moreover, it is often the case that one can relate an "outer" period of the corresponding residual Eisenstein series to the "inner" M_H-period. Thus $\ell_H(E_{-1}(\bullet, \varphi))$ is related to $\mathrm{res}_{s=s_0} L(s, \pi, \rho)$ for a certain subgroup H of G such that $H \cap M = M_H$, where $E_{-1}(\bullet, \varphi)$ is the residual Eisenstein series. Examples of such phenomena appear in [JR92b], [Jia98], [GRS99] — see also the discussion in the last section of the present article. In order to convert this fact into a spectral identity of the type implied by the trace formula, let us recall the definition of the **spherical distribution** of a representation Π distinguished by H. It is given by

$$\mathcal{B}^{\Pi}_{\ell_H, \overline{\ell_H}}(f) = \sum_{\{\varphi\}} \ell_H(\Pi(f)\varphi)\overline{\ell_H(\varphi)} \tag{1}$$

for any $f \in C_c^\infty(G(\mathbb{A}))$, where φ runs over an orthonormal basis of Π. In the case at hand Π is the representation given by the residual Eisenstein series from π. In order to fix ideas assume that π (and hence also Π) is everywhere unramified and let us ignore the test function f from our discussion. Then (1) reduced to a single summand corresponding to the unramified vector $E_{-1}(\bullet, \varphi_0)$ of L^2-norm 1. We have

$$(E_{-1}\varphi_0, E_{-1}\varphi_0) = (M_{-1}\varphi_0, \varphi_0) = m_{-1}\,(\varphi_0, \varphi_0)$$

where M_{-1} is the residue of the intertwining operator at s_0. Hence $\mathcal{B}_{\ell_H, \overline{\ell_H}}$ is essentially given by $(\mathrm{res}_{s=s_0} L(s, \pi, \rho))^2/m_{-1}$. On the other hand, by a well-known formula of Langlands ([Lan71])

$$m_{-1} = (\mathrm{res}_{s=s_0} L(s, \pi, \rho))/L(s_0 + 1, \pi, \rho).$$

Thus we get $(\mathrm{res}_{s=s_0} L(s, \pi, \rho))L(s_0+1, \pi, \rho)$. In view of the integral presentation of $L(s, \pi, \rho)$ this suggests that the spectral distribution for the group M to be compared to $\mathcal{B}^{\Pi}_{\ell_H, \overline{\ell_H}}(f)$ will be $\mathcal{B}^{\pi}_{\ell_{M_H}, \overline{\ell_{M_H}\cdot\mathfrak{E}}}(f')$, the so-called **weighted spherical distribution**, where

$$\ell_{M_H, \mathfrak{E}}(\varphi) = \int_{M_H \cap M(\mathbb{A})^1} \varphi(m)\mathfrak{E}(m, s_0 + 1)\, dm.$$

This comparison is our main result in this paper in the case where $G = Sp_{2n}$, H is the period subgroup $Sp_n \times Sp_n$ and $M = GL_{2n}$ is the Siegel Levi subgroup with $M_H = GL_n \times GL_n$. In this case, the relevant L-function product is $L(s, \pi)L(2s, \pi, \wedge^2)$ and we look at the pole at $s = \frac{1}{2}$. The integral presentation for that L-function was studied by Bump-Friedberg ([BF90]).* It is given by

$$\int\int \varphi((m_1, m_2))\mathfrak{E}(m_1, 4s - 1)|\det(m_1)|^{2s-1}\, dm_1\, dm_2$$

where $\mathfrak{E}(\bullet, s)$ is a degenerate Eisenstein series on GL_n and $m_1, m_2 \in GL_n(F)\backslash GL_n(\mathbb{A})$ are integrated over the subset $|\det(m_1 m_2)| = 1$. The transfer f' of f is given explicitly as a variant of Harish-Chandra's constant term map.

The comparison reflects the isomorphism of the symmetric space $Sp_n \times Sp_n\backslash Sp_{2n}/Sp_n \times Sp_n$ with $GL_n \times GL_n\backslash GL_{2n}/GL_n \times GL_n$ given by intersection with M. It is the cuspidal part of the hypothetical trace formula identity

$$\int_{(H(F)\backslash H(\mathbb{A}))^2} K_f(h_1, h_2)\, dh_1\, dh_2 = \int_{(M_H(F)\backslash M_H(\mathbb{A})\cap M(\mathbb{A})^1)^2} K_{f'}(m_1, m_2)\mathfrak{E}(m_2)\, dm_1\, dm_2 \tag{2}$$

* The subtle issues of ramified local L-factors and compatibility with Shahidi's definition are irrelevant for the discussion which follows.

where K_f ($K_{f'}$) is the usual automorphic kernel on G and M respectively. The left-hand side of (2) is a trace formula on a symmetric space. This notion was introduced and treated in [JLR93]. It is conjectured to compare with a certain "weighted" trace formula on a group. The identity (2) is very much in accordance with this conjecture.

The main ingredient in the proof, outside the discussion above, is an alternative formula for the period of a residual Eisenstein series in terms of its constant term. This expression involves the special automorphic form $\mathfrak{E}$. It is based on the computation of [AGR93]. In contrast, the approach taken in [JLR03] was somewhat special to GL_{2n} since it relied on the Fourier expansion of a residual Eisenstein series on G. Even in that case the new approach is simpler. The idea of using the method of [AGR93] was suggested to us by David Ginzburg. We are very grateful to him.

Let now E/F be a quadratic extension and let $U(2n) \subset GL_{2n}/E$ be a quasi-split unitary group with respect to an anti-Hermitian form. Thus $Sp_n = U(2n) \cap GL_{2n}(F)$ and $U(2n)$ contains GL_n/E as the Levi component of a parabolic subgroup. By a similar method we obtain a comparison between the pairs $(U(2n), Sp_n)$ and $(GL_n/E, GL_n/F)$. This is a non-split version of the case considered in [JLR03]. The representations of GL_n/E which are distinguished by GL_n/F were studied by Flicker ([Fli88]). They are the ones for which the Asai L-function has a pole at $s = 1$. The relation between the outer period of the residual Eisenstein series and the inner period will appear in the upcoming thesis of Tanai.

Going back to the general case considered before, there is a related, but simpler, trace formula identity

$$\int_{H\backslash H(\mathbb{A})} \int_{U_0\backslash U_0(\mathbb{A})} K_f(h,u)\psi(u)\,du\,dh = \int_{M_H\backslash M_H(\mathbb{A})^{(1)}} \int_{U_0^M\backslash U_0^M(\mathbb{A})} K_{f'}(m,u')\psi(u')\,du'\,dm$$

where U_0 is the maximal unipotent and ψ is a character of U_0 which is trivial on U and non-degenerate on $U_0 \cap M$. The geometric side of this identity was considered in [JR92a] and [JMR99]. Following [JLR03] the spectral part of this identity can be deduced every time we have a relation between "outer" and "inner" periods. In the last section we review some other known examples of such relations [Jia98], [GJ01]. We note, however, that the relation (2) may be more restrictive.

We mention that the property of being distinguished by M_H is related not only to the existence of a residual spectrum on a bigger group G, but also, perhaps more interestingly, to being a functorial image of yet another (not necessary algebraic) group G'.* Interestingly enough, the functoriality can sometimes be constructed explicitly (via the "descent" construction) from residual Eisenstein series on G (see [GRS99]). It would be very desirable to have a trace formula interpretation of this. This is the crux of a work in progress by Mao and Rallis.

2. Notation and preliminaries

2.1. Roots, weights and vector spaces. Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ its ring of adèles. By our convention, a group over F and its F-points will be denoted by the same letter. For any group X over F set $X(\mathbb{A})^1 = \bigcap \ker|\chi|$ where χ ranges over all rational characters of X and let $\delta_X(\bullet)$ be the modulus function on $X(\mathbb{A})$. Throughout let G be the group Sp_{2n} with $n \geq 1$, viewed as the subgroup of GL_{4n} preserving the skew symmetric form $[\bullet, \bullet]$ corresponding to

$$\epsilon_{4n} = \begin{pmatrix} 0 & w_{2n} \\ -w_{2n} & 0 \end{pmatrix}$$

where

$$w_{2n} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Let $\mathbb{V} = F^{4n}$ be the vector space of row vectors on which G acts on the right. Let T_0 be the maximal torus of G consisting of diagonal matrices. It is isomorphic to $(F^*)^{2n}$. Let $P_0 = T_0U_0$ be the standard Borel subgroup. The embedding

$$\mathbb{R} \hookrightarrow F \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow \mathbb{A}_F, \tag{3}$$

defined by $x \mapsto 1 \otimes x$, will be used to obtain a subgroup A_0 of $T_0(\mathbb{A})$ isomorphic to $(\mathbb{R}_+^*)^{2n}$. We let

$$\Delta_0 = \{\alpha_1, \dots, \alpha_{2n}\}$$

be the set of simple roots for G, in the usual ordering. Let $\hat{\Delta} = \{\varpi_1, \dots, \varpi_{2n}\}$ be the set of fundamental weights. As usual, $\mathfrak{a}_0$ will be the real vector space generated by the co-characters of T_0, and $\mathfrak{a}_0^*$ its dual. Explicitly $\mathfrak{a}_0^*$ is generated by ε_i, $i = 1, \dots, 2n$ where the character ε_i on T_0 is given by the i-th entry of the

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* In fact, the latter property seems to be more fundamental; for example, it may hold even when the M_H-period is not directly related to special values of L-functions.

diagonal matrix. The Weyl group $N_G(T_0)/T_0$ of G will be denoted by W_G. For any standard parabolic $P = M \cdot U$ of G we have the decomposition

$$\mathfrak{a}_0 = \mathfrak{a}_M \oplus \mathfrak{a}_0^M$$

and similarly for the dual spaces. We let Δ_0^M or Δ_0^P be the set of simple roots of T_0 in $U_0^M = U_0 \cap M$. Then $(\mathfrak{a}_0^M)^*$ is spanned by Δ_0^M and $\mathfrak{a}_M^*$ is spanned by the rational characters of M. Let H_0: $G(\mathbb{A}) \to \mathfrak{a}_0$ be the standard height function of G. On $T_0(\mathbb{A})$ it is given by

$$e^{\langle \chi, H_0(t) \rangle} = \prod_v |\chi_v(t_v)|$$

for any rational character χ of T_0. It extends to a left-$U_0(\mathbb{A})$-right-$\mathbf{K}$-invariant function on $G(\mathbb{A})$ by the Iwasawa decomposition, where $\mathbf{K}$ is the standard maximal compact subgroup of $G(\mathbb{A})$. Similarly, we have H_M: $G(\mathbb{A}) \to \mathfrak{a}_M$ for any Levi subgroup M. Let A_M be the intersection of A_0 with the center of $M(\mathbb{A})$. Then H_M defines an isomorphism between A_M and $\mathfrak{a}_M$. Let $X \mapsto e^X$ be its inverse. Let $\rho_P \in \mathfrak{a}_M^*$ be such that $\delta_P(p) = e^{\langle 2\rho_P, H_M(p) \rangle}$ for all $p \in P(\mathbb{A})$.

For any parabolic $P = M \cdot U$, an automorphic representation π of $M(\mathbb{A})$ and a parameter $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ we let $\mathcal{A}(U(\mathbb{A})M \backslash G(\mathbb{A}))_{\pi,\lambda}$ be the space of smooth functions on $G(\mathbb{A})$, left invariant under $U(\mathbb{A}) \cdot M$ such that for any $g \in G(\mathbb{A})$ the function $m \mapsto e^{-\langle \lambda + \rho_P, H_P(m) \rangle} \varphi(mg)$ belongs to the space of π. We will always assume that π is trivial on A_M and will denote the action of $C_c^\infty(G(\mathbb{A}))$ by $I(f, \lambda)$. Set $\mathcal{A}(U(\mathbb{A})M \backslash G(\mathbb{A}))_\pi = \mathcal{A}(U(\mathbb{A})M \backslash G(\mathbb{A}))_{\pi,0}$. The space $\mathcal{A}(U(\mathbb{A})M \backslash G(\mathbb{A}))_{\pi,\lambda}$ is isomorphic to (the smooth part of) $\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi \otimes e^{\langle \lambda, H_M(\cdot) \rangle} = I(\pi, \lambda)$. This applies in particular to the unit representation which we will denote by $\mathbf{1}$. We will also denote by $\mathcal{A}^c(U(\mathbb{A})M \backslash G(\mathbb{A}))$ the space of cuspidal automorphic forms on $U(\mathbb{A})M \backslash G(\mathbb{A})$ such that $\varphi(ag) = e^{\langle \rho_P, H_P(a) \rangle} \varphi(g)$ for all $a \in A_M$.

2.2. The setup. Let H be the group $Sp_n \times Sp_n$ viewed as a subgroup of G. Explicitly, let $\mathbb{U}$ be the $2n$-dimensional subspace of $\mathbb{V}$ defined by

$$x_2 = x_4 = \cdots = x_{2n} = 0 = x_{2n+1} = x_{2n+3} = \cdots = x_{4n-1}.$$

Then $[\bullet, \bullet]|_{\mathbb{U}}$ is non-degenerate and H is the stabilizer of $\mathbb{U}$. The map

$$h \mapsto (h|_{\mathbb{U}}, h|_{\mathbb{U}^\perp}) \tag{4}$$

defines an isomorphism of H with $Sp_{\mathbb{U}} \times Sp_{\mathbb{U}^\perp} \simeq Sp_n \times Sp_n$. The notation for H will be similar to that of G, except that it will usually be appended by H. For any subgroup X of G we denote by X_H the intersection of X with H. Since T_0 is contained in H we can identify $\mathfrak{a}_0^H$ with $\mathfrak{a}_0$ and similarly for the dual spaces. The group $(P_0)_H = T_0 \cdot (U_0)_H$ is a Borel subgroup of H. The restriction of H_0 to $H(\mathbb{A})$ is the height function with respect to $H(\mathbb{A})$.

From now on $P = M \cdot U$ will denote the Siegel parabolic in G with its standard Levi decomposition and $Q = L \cdot V$ will be the "Heisenberg" parabolic. Concretely, P is the stabilizer of the $2n$-dimensional isotropic subspace $\mathbb{V}'$ of $\mathbb{V}$ of vectors whose first $2n$ coordinates are zero, and Q is the stabilizer of the line $F \cdot v_0$ where $v_0 = (0, \dots, 0, 1)$. We have $P_H = M_H \cdot U_H$ and $Q_H = L_H \cdot V_H$. Under the map (4), P_H is isomorphic to the product of Siegel parabolic subgroups of $Sp_{\mathbb{U}}$ and $Sp_{\mathbb{U}^\perp}$, namely the stabilizers of $\mathbb{U} \cap \mathbb{V}'$ and $\mathbb{U}^\perp \cap \mathbb{V}'$ respectively. Also, M_H is isomorphic to $GL_{\mathbb{U}\cap\mathbb{V}'} \times GL_{\mathbb{U}^\perp\cap\mathbb{V}'} = GL_n \times GL_n$, and Q_H is a maximal parabolic of H which is of the form $Q' \times Sp_n$ where Q' is a Heisenberg parabolic of Sp_n.

The weight $\varpi_{2n} \in \mathfrak{a}_M^*$ corresponds to the character

$$m \mapsto |\det_{\mathbb{V}'}(m)|^{-1}.$$

We will sometimes identify $\mathfrak{a}_M^*$ with $\mathbb{R}$ by $s\frac{\varpi_{2n}}{2} \mapsto s$. Under this identification $\rho_P = 2n+1$ and $\rho_{P_H} = n+1$. Finally, let $Q_1 = P \cap Q_H = Q \cap P_H$ and let $P_1 = Q_1 \cap M_H = Q \cap M_H$. Then Q_1 is a parabolic subgroup of H and P_1 is a maximal parabolic of M_H. The groups P_1, Q_1 have the same standard Levi factor, while their unipotent radicals differ by U_H.

The convention about Haar measures will be the following. On any discrete group we take the counting measure. For any unipotent group N we take the Tamagawa measure so that $\mathrm{vol}(N\backslash N(\mathbb{A})) = 1$. On $\mathbf{K}$ we take the measure of total mass 1. We fix a Haar measure dg on $G(\mathbb{A})$. The Haar measure on $M(\mathbb{A})$ will be determined by

$$\int_{G(\mathbb{A})} f(g)\,dg = \int_{\mathbf{K}} \int_{U(\mathbb{A})} \int_{M(\mathbb{A})} f(muk)\,dm\,du\,dk.$$

The measure on $\mathfrak{a}_M$ will be the pull-back of dx under $X \mapsto \langle \frac{\varpi_{2n}}{2}, X \rangle$. This will define a Haar measure on $M(\mathbb{A})^1$ by the isomorphism $M(\mathbb{A})/M(\mathbb{A})^1 \simeq \mathfrak{a}_M$. Similarly, we choose a Haar measure dh on $H(\mathbb{A})$ and choose the compatible Haar measure on $M_H(\mathbb{A})$ with respect to the Iwasawa decomposition relative to P_H, where on the maximal compact $\mathbf{K}_H = \mathbf{K} \cap H(\mathbb{A})$ we take the measure of total mass 1. We set $M_H(\mathbb{A})^{(1)} = M_H(\mathbb{A}) \cap M(\mathbb{A})^1$ and endow it with a measure defined through the isomorphism $M_H(\mathbb{A})/M_H(\mathbb{A})^{(1)} \simeq \mathfrak{a}_M$. On the idèles $\mathbb{I}_F$ we take the unnormalized Tamagawa measure. The measure on $\mathbb{I}_F^1$ will be taken so that the quotient measure on $\mathbb{I}_F/\mathbb{I}_F^1$ will be the pull-back of d^*t under $|\bullet|_F$.

2.3. Eisenstein series. We will consider various Eisenstein series. For any $\varphi \in \mathcal{A}^c(U(\mathbb{A})M\backslash G(\mathbb{A}))$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ we consider the Eisenstein series $E(g,\varphi,\lambda)$ on $G(\mathbb{A})$ which is given by the series

$$\sum_{\gamma \in P\backslash G} \varphi(\gamma g) e^{\langle \lambda, H_P(\gamma g)\rangle}$$

in the range of absolute convergence. By general theory (e.g. [MW95]) the residue $E_{-1}(g,\varphi)$ of $E(g,\varphi,s)$ at $s=1$ is a (possibly zero) square integrable function on $G\backslash G(\mathbb{A})$. The only possible non-zero constant term is along P and it is given by $M_{-1}\varphi$ where M_{-1} is the residue of the intertwining operator at 1. The inner product of two residual Eisenstein series is given by

$$\int_{G\backslash G(\mathbb{A})} E_{-1}(g,\varphi_1)\overline{E_{-1}(g,\varphi_2)}\, dg = \int_{\mathbf{K}} \int_{M\backslash M(\mathbb{A})^1} M_{-1}\varphi_1(mk)\overline{\varphi_2(mk)}\, dm\, dk.$$

Let now $\mathbb{W}$ be an m-dimensional vector space over F. The (normalized) degenerate Eisenstein series on $GL_{\mathbb{W}}$ was studied in [JS81]. It is defined by

$$\mathcal{E}^{\mathbb{W}}_{\Phi}(g,s) = |\det(g)|^{(s+1)/2} \cdot \int_{F^*\backslash \mathbb{I}_F} \sum_{v \in \mathbb{W}-\{0\}} \Phi(t\cdot vg)|t|^{m(s+1)/2}\, d^*t$$

where $g \in GL_{\mathbb{W}}(\mathbb{A})$ and $\Phi \in \mathcal{S}(\mathbb{W}(\mathbb{A}))$ is a Schwartz–Bruhat function on $\mathbb{W}(\mathbb{A})$. The series is absolutely convergent for $\mathrm{Re}(s) > 1$ and, following the method of Tate's thesis, it admits a meromorphic continuation with poles at ± 1 and a functional equation relating s and $-s$. In any vertical strip there exists c, N such that

$$(s^2-1)|\mathcal{E}^{\mathbb{W}}_{\Phi}(g,s)| \le c||g||^N$$

for all g.

In particular, for any $\Phi \in \mathcal{S}(\mathbb{U}(\mathbb{A}))$ let $\mathcal{E}_\Phi(\bullet,s)$ be the restriction of $\mathcal{E}^{\mathbb{U}}_\Phi(\bullet,s)$ to $Sp_{\mathbb{U}}(\mathbb{A})$. We shall view $\mathcal{E}_\Phi(\bullet,s)$ as an automorphic form on $H(\mathbb{A})$ which depends only on the $Sp_{\mathbb{U}}$ coordinate. The residue of $\mathcal{E}_\Phi(\bullet,s)$ at $s=1$ is the constant function $\lambda_{-1}/n \cdot \int_{\mathbb{U}(\mathbb{A})} \Phi(x)\, dx$ where $\lambda_{-1} = \mathrm{vol}(F^*\backslash \mathbb{I}^1_F)$. We may also write

$$\mathcal{E}_\Phi(h,s) = \sum_{\gamma \in Q_H\backslash H} \phi_{\Phi,s}(\gamma h)$$

where $\phi_{\Phi,s} \in \mathcal{A}(V_H(\mathbb{A})L_H\backslash H(\mathbb{A}))_{\mathbf{1},s\rho_{Q_H}}$ is defined by

$$\phi_{\Phi,s}(h) = \int_{\mathbb{I}_F} \Phi(t\cdot v_0 h)|t|^{n(s+1)}\, d^*t.$$

We denote by $\mathfrak{E}(\bullet, s)$ the function $\mathcal{E}_{\Phi_0}^{\mathbb{U}\cap\mathbb{V}'}(\bullet, s)$ on $M_H\backslash M_H(\mathbb{A})$ (depending only on the $Sp_{\mathbb{U}}$ part) corresponding to the "standard" $\Phi_0 \in \mathcal{S}(\mathbb{U}(\mathbb{A}) \cap \mathbb{V}'(\mathbb{A}))$. (By the latter we mean the factorizable function which is the characteristic function of the standard lattice at the non-archimedean places, and it is the normalized Gaussian at the archimedean places.) For any $\Phi \in \mathcal{S}(\mathbb{U}(\mathbb{A}))$ we let

$$\mathbb{E}_\Phi(h, s) = \mathcal{E}_{\Phi_1}^{\mathbb{U}\cap\mathbb{V}'}(e, s)$$

where Φ_1 is the restriction of $\Phi(\bullet h)$ to $\mathbb{U}(\mathbb{A}) \cap \mathbb{V}'(\mathbb{A})$. We have

$$\mathbb{E}_\Phi(h, s) = \sum_{Q_1\backslash P_H} \phi_{\Phi,s'}(\gamma h) = \sum_{P_1\backslash M_H} \phi_{\Phi,s'}(\gamma h) \tag{5}$$

where $s' = (s-1)/2$, since $\dim(\mathbb{U}\cap\mathbb{V}') = \frac{1}{2}\dim\mathbb{U}$. The function $\mathbb{E}_\Phi(h, s)$ depends only on the $Sp_{\mathbb{U}}$ coordinate of h and as a function on $Sp_{\mathbb{U}}$ it belongs to $\mathcal{A}(U'(\mathbb{A})M'\backslash Sp_{\mathbb{U}}(\mathbb{A}))$ where $P' = U'M'$ is the Siegel parabolic subgroup of $Sp_{\mathbb{U}}$ stabilizing $\mathbb{U}\cap\mathbb{V}'$. Under left multiplication by the center $Z_{M'}(\mathbb{A})$ of $M'(\mathbb{A})$, $\mathbb{E}_\Phi(\bullet, s)$ transforms according to the character $|\det_{\mathbb{U}\cap\mathbb{V}'}|^{-(s'+1)}$.

3. The H-period

Our first main Theorem is the following

THEOREM 1: *For any $\varphi \in \mathcal{A}^c(U(\mathbb{A})M\backslash G(\mathbb{A}))$ and $\Phi \in \mathcal{S}(\mathbb{U}(\mathbb{A}))$ we have*

$$\frac{\lambda_{-1}}{n}\int_{\mathbb{U}(A)} \Phi(x)\,dx \cdot \int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\,dh = \int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^{(1)}} M_{-1}\varphi(mk)\mathbb{E}_\Phi(mk, 3)\,dm\,dk.$$

In particular, taking Φ to be the "standard" $\mathbf{K}_H$-invariant function on $\mathbb{U}(\mathbb{A})$,

$$\frac{\lambda_{-1}}{n}\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\,dh = \int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^{(1)}} M_{-1}\varphi(mk)\mathfrak{E}(m,3)\,dm\,dk.$$

Remark: The alternative formula

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\,dh = \int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^{(1)}} \varphi(mk)\,dm\,dk \tag{6}$$

was proved in [GRS99].

Here is a possible interpretation of Theorem 1. Let π be a cuspidal representation of $M(\mathbb{A})$, trivial on A_M. As in ([JLR03]) for any $\psi \in \mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,-1}$ the form

$$\Phi \in \mathcal{S}(\mathbb{U}(\mathbb{A})) \mapsto \int_{A_M U_H(\mathbb{A}) P_H\backslash H(\mathbb{A})} \psi(h)\,\mathbb{E}_\Phi(h,3)\,dh$$

is well defined and factors through $\tau = \mathrm{Ind}_{Q_H(\mathbb{A})}^{H(\mathbb{A})} \delta_{Q_H}^{1/2}$ of $H(\mathbb{A})$. Thus we get an $H(\mathbb{A})$-equivariant map

$$\Upsilon\colon \mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,-1} \longrightarrow \tau^\vee.$$

The Theorem implies that the image of $\Upsilon \circ M_{-1}$ is the identity subrepresentation of $\tau^\vee$.

4. An identity of Bessel distributions

As in [JLR03] Theorem 1 and the alternative formula (6) can be used to give an identity of Bessel distributions. As mentioned in the Introduction, this identity is the cuspidal part of the spectral side of a hypothetical identity

$$\int_{H\backslash H(\mathbb{A})} \int_{H\backslash H(\mathbb{A})} K_f(h_1,h_2)\, dh_1\, dh_2 \\ = \int_{M_H\backslash M_H(\mathbb{A})^{(1)}} \int_{M_H\backslash M_H(\mathbb{A})^{(1)}} K_{f'}(m_1,m_2)\mathfrak{E}(m_2,3)\, dm_1\, dm_2$$

(suitably regularized to overcome convergence issues). However, we obtain the identity without appealing to the trace formula.

Let π be a cuspidal automorphic representation of $M(\mathbb{A})$. We consider the representation Π of $G(\mathbb{A})$ spanned by $E_{-1}(\cdot,\varphi)$, $\varphi \in \mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_\pi$. It lies in the discrete spectrum of $L^2(G\backslash G(\mathbb{A}))$. Recall that the **Bessel distribution** of Π with respect to linear forms l_1, l_2 is defined by

$$\mathcal{B}^{\Pi}_{l_1,\overline{l_2}}(f) = \sum_{\varphi_i} l_1(\Pi(f)\varphi_i)\overline{l_2(\varphi_i)}$$

for $f \in C_c^\infty(G(\mathbb{A}))$ where φ_i ranges over an orthonormal basis of Π consisting of smooth functions. Similarly, for π (where the unitary structure is given by the inner product on $M\backslash M(\mathbb{A})^1$).

Let ℓ_H be the linear form on Π defined by

$$\ell_H(\varphi) = \int_{H\backslash H(\mathbb{A})} \varphi(h)\, dh$$

and let ℓ_{M_H} be the period over $M_H\backslash M_H(\mathbb{A})^{(1)}$ as a linear form on π. Let also $\ell_{M_H,\mathfrak{E}}$ be the linear form on π defined by

$$\ell_{M_H,\mathfrak{E}}(\varphi) = \int_{M_H\backslash M_H(\mathbb{A})^{(1)}} \varphi(m)\mathfrak{E}(m,3)\, dm.$$

THEOREM 2: *We have*

$$\frac{\lambda_{-1}}{n} \cdot \mathcal{B}^{\Pi}_{\ell_H, \bar{\ell}_H}(f) = \mathcal{B}^{\pi}_{\ell_{M_H}, \bar{\ell}_{M_H}, \mathfrak{e}}(f'_{\mathbf{K}_H}) \tag{7}$$

where $f'_{\mathbf{K}_H}$ is the function on $M(\mathbb{A})$ defined by

$$f'_{\mathbf{K}_H}(m) = e^{\langle \frac{1}{2}\varpi_{2n} + \rho_P, H_M(m) \rangle} \cdot \int_{\mathbf{K}_H} \int_{\mathbf{K}_H} \int_{U(\mathbb{A})} f(k'muk)\, du\, dk'\, dk.$$

Note that $f'_{\mathbf{K}_H}$ is essentially Harish-Chandra's constant term map from G to M.

The proof of Theorem 2 is identical to that of Theorem 3 of [JLR03] and will be omitted.

We can also state a local analogue of Theorem 2 obtained through the factorization of the integrals $\int_{M_H \cap M_H(\mathbb{A})^{(1)}} \phi(m)\mathbb{E}(m,s)\, dm$ considered in [BF90].

5. A distributional formula

We recall that the formula (6) was proved by approximating the residual Eisenstein series. The smooth way to do it is using pseudo-Eisenstein series. In order to prove Theorem 1 we will also need to approximate the constant function on $H(\mathbb{A})$ appropriately.

5.1. PSEUDO-EISENSTEIN SERIES (cf. [MW95]). Let φ be a smooth $\mathbf{K}$-finite function on $U(\mathbb{A})M\backslash G(\mathbb{A})$ such that for all $k \in \mathbf{K}$ the function $m \mapsto \varphi(mk)$ on $M(\mathbb{A})^1$ is cuspidal and for all $g \in G(\mathbb{A})$ the function $a \mapsto \varphi(ag)$ is compactly supported on A_M. The right action of $G(\mathbb{A})$ on this space gives rise to an action of $C_c^\infty(G(\mathbb{A}))$ which will be denoted by $R(f)$. (We will use the same notation for the right regular representation of $G(\mathbb{A})$ on functions on $G\backslash G(\mathbb{A})$.) From φ we build the pseudo-Eisenstein series

$$\Theta_\varphi(g) = \sum_{\gamma \in P\backslash G} \varphi(\gamma g).$$

We set

$$\hat{\varphi}_s(g) = \int_{\mathfrak{a}_M} e^{-\langle s\frac{\varpi_{2n}}{2} + \rho_P, X \rangle} \varphi(e^X g)\, dX.$$

Thus, $\hat{\varphi} \in \mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,s}$ and for any s_0

$$\varphi(g) = \frac{1}{2\pi i} \int_{\operatorname{Re} s = s_0} \hat{\varphi}_s(g)\, ds.$$

It follows that

$$\Theta_\varphi(g) = \frac{1}{2\pi i} \int_{\operatorname{Re} s>>0} E(g, \hat{\varphi}_s, s)\, ds.$$

All the constant terms of Θ_φ vanish except the one along U. The latter is given by

$$\Theta_\varphi^U(h) = \varphi(h) + M\varphi(h), \tag{8}$$

where

$$M\varphi(h) = \int_{U(\mathbb{A})} \varphi(wuh)\, dh$$

and w is the longest element of the Weyl group. We have

$$\widehat{M\varphi}_{-s} = M(s)\hat{\varphi}_s \tag{9}$$

for $\operatorname{Re} s >> 0$ where

$$M(s)\colon \mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,s} \to \mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,-s}$$

is the usual intertwining operator.

Similarly, to approximate the constant function on $H(\mathbb{A})$ we define for any holomorphic function $\sigma \in \mathcal{P}(\mathbb{C})$ of Paley–Wiener type

$$\theta_{\Phi,\sigma}(h) = \int_{\operatorname{Re}(s)>>0} \sigma(s)\mathcal{E}_\Phi(h,s)\, ds. \tag{10}$$

We may write

$$\theta_{\Phi,\sigma}(h) = \sum_{\gamma\in Q_H\backslash H} \mathcal{F}_{\Phi,\sigma}(\gamma h),$$

where $\mathcal{F}_{\Phi,\sigma}$ is the function on $Q_H(\mathbb{A})^1\backslash H(\mathbb{A})$ defined by

$$\mathcal{F}_{\Phi,\sigma}(h) = \int_{\mathrm{I}_F} \Phi(t\cdot v_0 h)\mathcal{C}_\sigma(\log(|t|^n))\, d^*t$$

and $\mathcal{C}_\sigma$ is the smooth compactly supported function on $\mathbb{R}$ given by

$$\mathcal{C}_\sigma(X) = \int_{i\mathbb{R}} \sigma(s)e^{(s+1)X}\, ds.$$

(The contour of integration can be shifted to any $\operatorname{Re}(s) = s_0$.) We may recover σ from $\mathcal{C}_\sigma$ by

$$\sigma(s) = \frac{1}{2\pi i}\int_{\mathbb{R}} \mathcal{C}_\sigma(X)e^{-(s+1)X}\, dX.$$

Set

$$\mathfrak{P}_\Phi(\sigma,\varphi) = \int_{H\backslash H(\mathbb{A})} \Theta_\varphi(h)\theta_{\Phi,\sigma}(h)\, dh.$$

We also fix $f \in C_c^\infty(G(\mathbb{A}))$. The main assertion is the following

THEOREM 3: *The following two formulas hold for $\mathfrak{P}_\Phi(\sigma, R(f)\varphi)$ (for any $0 < s_0 < 1$):*

$$\begin{aligned}
&\mathfrak{P}_\Phi(\sigma,R(f)\varphi)\\
&=2\pi i \cdot \sigma(1)\frac{\lambda_{-1}}{n}\int_{\mathbb{U}(A)}\Phi(x)\,dx\int_{H\backslash H(\mathbb{A})}R(f)\Theta_\varphi(h)\,dh\\
&\quad+\int_{\mathrm{Re}\,s=s_0}\sigma(s)\int_{H\backslash H(\mathbb{A})}R(f)\Theta_\varphi(h)\mathcal{E}_\Phi(h,s)\,dh\,ds\\
&=2\pi i\cdot\sigma(1)\int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^{(1)}}\mathbb{E}_\Phi(mk,3)I(f,-1)M_{-1}\hat{\varphi}_1(mk)\,dm\,dk\\
&\quad+\int_{\mathrm{Re}\,s=s_0}\sigma(s)\int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^{(1)}}\mathbb{E}_\Phi(mk,2s+1)I(f,-s)\\
&\qquad\qquad[\hat{\varphi}_{-s}(mk)+M(s)\hat{\varphi}_s(mk)](mk)\,dm\,dk\,ds.
\end{aligned}$$

We will prove Theorem 3 in the next section. Let us show that it implies Theorem 1. Fix f, φ and view Theorem 3 as a distributional identity in σ. We may separate the "atomic" part from the continuous one (cf. [JLR03, Lemma 3]) to obtain

$$\begin{aligned}
&\frac{\lambda_{-1}}{n}\int_{\mathbb{U}(A)}\Phi(x)\,dx\int_{H\backslash H(\mathbb{A})}R(f)\Theta_\varphi(h)\,dh\\
&\quad=\int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^{(1)}}\mathbb{E}_\Phi(mk,3)M_{-1}[I(f,1)\hat{\varphi}_1](mk)\,dm\,dk.
\end{aligned}\tag{11}$$

On the other hand, by shifting the contour of integration and interchanging the order of integration we also have

(12)

$$\begin{aligned}
&\int_{H\backslash H(\mathbb{A})}R(f)\Theta_\varphi(h)\,dh=\frac{1}{2\pi i}\int_{H\backslash H(\mathbb{A})}\int_{\mathrm{Re}\,s>>0}E(h,I(f,s)\hat{\varphi}_s,s)\,ds\,dh\\
&=\int_{H\backslash H(\mathbb{A})}E_{-1}(h,I(f,1)\hat{\varphi}_1)\,dh+\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=s_0}\int_{H\backslash H(\mathbb{A})}E(h,I(f,s)\hat{\varphi}_s,s)\,dh\,ds
\end{aligned}$$

for any $0 < s_0 < 1$. This is justified by the following Lemma which will be proved below.

LEMMA 1: *Let $\phi \in \mathcal{A}^c(U(\mathbb{A})M\backslash G(\mathbb{A}))$. Then the integral*

$$\int_{H\backslash H(\mathbb{A})}E(h,I(f,s)\phi,s)\,dh\tag{13}$$

is absolutely convergent for $0 < \mathrm{Re}(s) < 1$. Moreover, for any fixed $0 < s_0 < 1$

$$\sup_{\mathrm{Re}(s)=s_0}\int_{H\backslash H(\mathbb{A})}|E(h,I(f,s)\phi,s)|\,dh<\infty.$$

Granted the Lemma, we use once again Lemma 3 of [JLR03] (this time with respect to φ) to conclude from (11) and (12) the assertion of Theorem 1 (as well as the vanishing of (13)) with $I(f,1)\hat{\varphi}_1$ instead of φ. It remains to invoke the Dixmier–Malliavin Theorem ([DM78]).

To prove the Lemma, let $\mathcal{S}$ be a Siegel set for $G(\mathbb{A})$ of the form $\omega \times A_0(c_0) \times \mathbf{K}$, where ω is a certain compact subset of $P_0(\mathbb{A})^1$ and

$$A_0(c_0) = \{a \in A_0 : \langle \alpha, H_0(a)\rangle > c_0 \text{ for all } \alpha \in \Delta_0\}.$$

Since E^U is the only non-zero constant term $E(g,\phi,s) - E^U(g,\phi,s)$ is rapidly decreasing on $\mathcal{S}$. Note that $\mathcal{S} \cap H(\mathbb{A})$ is not a Siegel set for $H(\mathbb{A})$. However, we may choose $\mathcal{S}$ so that $\bigcup_{w\in W_M} w\mathcal{S}$ contains a Siegel set $\mathcal{S}^H$ of $H(\mathbb{A})$ where W_M is the Weyl group of M. Thus, $E(g,\phi,s) - E^U(g,\phi,s)$ is rapidly decreasing on $\mathcal{S}^H$ as well. Thus, to prove the first part it is enough to show that

$$\int_{\mathcal{S}^H} |E^U(h, I(f,s)\phi, s)| \, dh$$

converges. Since

$$E^U(I(f,s)\phi,s) = I(f,s)\phi(g)e^{\langle s\frac{\varpi_{2n}}{2}, H_P(g)\rangle} + M(s)I(f,s)\phi(g)e^{\langle -s\frac{\varpi_{2n}}{2}, H_P(g)\rangle}$$

and $I(f,s)\phi$ and $M(s)I(f,s)\phi$ are bounded (in fact, rapidly decreasing) on $M(\mathbb{A})^1 \cdot \mathbf{K}$, it will suffice to show the convergence of

$$\int_{\mathbf{K}_H}\int_{A_0^H(c_0)}\int_{\omega_H} e^{-\langle 2\rho_0^H, H_0(a)\rangle} e^{\langle \pm s\frac{\varpi_{2n}}{2}+\rho_P, H_P(a)\rangle} \, dp \, da \, dk.$$

This follows from the easily checked fact that $s_0\frac{\varpi_{2n}}{2} + \rho_P - 2\rho_0^H$ lies in the obtuse chamber relative to the roots of H.

To examine the dependence on s we need to control the decay of $E(g,\phi,s) - E^U(g,\phi,s)$ on $\mathcal{S}$ uniformly for $\operatorname{Re} s = s_0$, and similarly, the decay of $I(f,s)\varphi$ and $M(s)I(f,s)\varphi$ on $\mathcal{S}^M$. By [MW95, Corollary I.2.11] this will follow from the following Proposition.

Proposition 1: *Fix $0 < s_0 < 1$. Then there exist constants $c, n > 0$ such that*

$$|E(g, I(f,s)\phi, s)|, |I(f,s)\phi(g)|, |M(s)I(f,s)\phi(g)| \le c||g||^n$$

for all $g \in G(\mathbb{A})$ and $\operatorname{Re}(s) = s_0$.

Proof: As in [Lap] we can choose T depending polynomially on $\log(||g||)$, as well as on the support of f, so that

$$E(g, I(f,s)\phi, s) = \int_{G(\mathbb{A})} f(x)E(gx,\phi,s)\,dx = \int_{G(\mathbb{A})} f(x)\Lambda^T E(gx,\phi,s)\,dx$$

$$= \int_{G(\mathbb{A})} f(g^{-1}x)\Lambda^T E(x,\phi,s)\,dx = \int_{G\backslash G(\mathbb{A})} \sum_{\gamma\in G} f(g^{-1}\gamma x)\Lambda^T E(x,\phi,s)\,dx.$$

By [MW95, I.2.4] we can find $c, N > 0$ such that

$$\sum_{\gamma\in G} f(g^{-1}\gamma x) \le c||g||^N$$

for all $x \in G(\mathbb{A})$. Hence,

$$|E(g, I(f,s)\phi, s)| \le c\cdot \mathrm{vol}(G\backslash G(\mathbb{A}))^{\frac{1}{2}} \cdot ||g||^N ||\Lambda^T E(\cdot,\phi,s)||_2. \tag{14}$$

By the Maass–Selberg relations ([Art80, §4]) $||\Lambda^T E(\cdot,\phi,s)||_2^2$ equals

$$\frac{e^{2T\cdot \mathrm{Re}\, s}}{2\,\mathrm{Re}\, s}||\phi||^2 + \mathrm{Re}\left[\frac{e^{(s-\bar{s})T}}{s-\bar{s}}(M(s)\phi,\phi)\right] - \frac{e^{-2T\cdot\mathrm{Re}\, s}}{2\,\mathrm{Re}\, s}||M(s)\phi||^2.$$

It is well-known that the operator $M(s)$ is bounded uniformly on $\mathrm{Re}(s) \ge 0$ away from its poles (e.g. [HC68]). Thus, as long as $\mathrm{Re}\, s = s_0$ and $|\Im s| > 1$, $||\Lambda^T E(\cdot,\phi,s)||_2$ is bounded by a constant multiple of e^T. The first inequality follows. Similarly, $M(s)I(f,s)\phi(g)$ equals

$$\begin{aligned}
&e^{\langle s\frac{\varpi_{2n}}{2}, H_P(g)\rangle}\int_{G(\mathbb{A})} f(x)M(s)\phi(gx)e^{\langle -s\frac{\varpi_{2n}}{2}, H_P(gx)\rangle}\,dx\\
&\quad= e^{\langle s\frac{\varpi_{2n}}{2}, H_P(g)\rangle}\int_{G(\mathbb{A})} f(g^{-1}x)M(s)\phi(x)e^{\langle -s\frac{\varpi_{2n}}{2}, H_P(x)\rangle}\,dx\\
&\quad= e^{\langle s\frac{\varpi_{2n}}{2}, H_P(g)\rangle}\int_{\mathbf{K}}\int_{M\backslash M(\mathbb{A})^1} M(s)\phi(mk)\\
&\qquad\left[\int_{U(\mathbb{A})}\int_{A_M}\sum_{\gamma\in M} f(g^{-1}ua\gamma mk)e^{\langle -s\frac{\varpi_{2n}}{2}-\rho_P, H_P(a)\rangle}\,da\,du\right]dm\,dk.
\end{aligned}$$

Since

$$\int_{U(\mathbb{A})}\int_{A_M}\sum_{\gamma\in M} f(g^{-1}ua\gamma mk)e^{\langle -s\frac{\varpi_{2n}}{2}-\rho_P, H_P(a)\rangle}\,da\,du$$

is bounded polynomially in $||g||$, the other inequality follows as before from Cauchy-Schwartz and the boundedness of $||M(s)\phi||_2$. The inequality involving ϕ itself is similar, but easier. ∎

6. Proof of Theorem 3

The first part of Theorem 3 follows by using (10) and shifting the contour of integration.

To prove the second part we first claim the following.

LEMMA 2: *Let ψ be a fixed non-trivial character of $F\backslash\mathbb{A}_F$. Denote by $\mathcal{W}^{U_0^M,\psi}$ the Fourier coefficient along the character $\psi(x_{1,2}+\ldots+x_{2n-1,2n})$ of $U_0^M(\mathbb{A})$. Then*

$$\mathfrak{P}_\Phi(\sigma,R(f)\varphi)=\int_{Z_M U_0^H(\mathbb{A})\backslash H(\mathbb{A})}\mathcal{W}^{U_0^M,\psi}\Theta_{(R(f)\varphi)^U}(h)\mathcal{F}_\sigma(h)\,dh. \tag{15}$$

Proof: We will prove the Lemma using the method of [AGR93, §4]. In fact, in [loc. cit.] the periods $\int_{H\backslash H(\mathbb{A})}\varphi'(h)\theta_{\Phi,\sigma}(h)\,dh$ are shown to be zero for cusp forms. However, much of the argument is still valid if φ' is assumed to have vanishing constant term along all parabolic subgroups which are not of Siegel type. To be precise, let P_i be the parabolic subgroup whose Levi part is $GL(1)^i\times Sp(2n-i)$, which we write as $T_i\times M_i$. Let U_i denote the unipotent radical of P_i. Let S_i be the stabilizer of (U_{i-1},ψ) in P_i. Then S_i contains U_i and its projection to the Levi part of P_i is $Z_i\times M_i\simeq F^*\times Sp(2n-i)$. Then the argument in [loc. cit.] shows (by induction) that

$$\mathfrak{P}_\Phi(\sigma,R(f)\varphi)=\int_{(S_i)_H\backslash H(\mathbb{A})}\mathcal{W}^{U_{i-1},\psi}\Theta_{R(f)\varphi}(h)\mathcal{F}_\sigma(h)\,dh \tag{16}$$

for $i=1,\ldots,2n$ provided that the right-hand side is absolutely convergent. We infer the Lemma from the case $i=n$. To prove the convergence, we argue as in [JS90] to get

$$\begin{aligned}
&\mathcal{W}^{U_{i-1},\psi}[R(f)\Theta_\varphi](utmk)\\
&\quad=\int_{G(\mathbb{A})}\mathcal{W}^{U_{i-1},\psi}\Theta_\varphi(utmkx)f(x)\,dx\\
&\quad=\int_{G(\mathbb{A})}\mathcal{W}^{U_{i-1},\psi}\Theta_\varphi(utmx)f(k^{-1}x)\,dx\\
&\quad=\int_{U_{i-1}(\mathbb{A})\backslash G(\mathbb{A})}\int_{U_{i-1}(\mathbb{A})}\mathcal{W}^{U_{i-1},\psi}\Theta_\varphi(utmu'x)f(k^{-1}u'x)\,du'dx\\
&\quad=\int_{U_{i-1}(\mathbb{A})\backslash G(\mathbb{A})}\mathcal{W}^{U_{i-1},\psi}\Theta_\varphi(utmx)\int_{U_{i-1}(\mathbb{A})}f(k^{-1}u'x)\;\psi(tu't^{-1})\,du'dx
\end{aligned}$$

for $u\in U_i(\mathbb{A})$, $t\in T_i(\mathbb{A})$, $m\in M_i(\mathbb{A})$, $k\in\mathbf{K}$. Since Θ_φ is bounded, $\mathcal{W}^{U_{i-1},\psi}R(f)\Theta_\varphi(utmk)$ is majorized by

$$\int_{U_{i-1}(\mathbb{A})\backslash G(\mathbb{A})}\left|\int_{U_{i-1}(\mathbb{A})}f(k^{-1}u'x)\;\psi(tu't^{-1})\,du'\right|dx.$$

The outer integral has compact support while the inner integral can be viewed as the Fourier transforms of a compact family of Schwartz functions in the entries of u'. Thus,

$$|\mathcal{W}^{U_{i-1},\psi}R(f)\Theta_\varphi(utmk)| \leq \Phi_0(t_1/t_2,\dots,t_{i-1}/t_i)$$

for some $\Phi_0 \in \mathcal{S}(\mathbb{A}_F^{i-1})$. Moreover, from the definition, it follows that

$$|\mathcal{F}_\sigma(utmk)| \leq \int_{F^*\backslash\mathbb{I}_F} \sum_{\gamma\in F^*} |\Phi(\gamma x\cdot v_0 tk)\mathcal{C}_\sigma(\log(|x|^n))|\, d^*x \leq \sum_{\gamma\in F^*}\phi_0(\gamma t_1^{-1})$$

for some some $\phi_0 \in \mathcal{S}(\mathbb{A}_F)$. Thus,

$$\begin{aligned}
&\int_{(S_i)_H\backslash H(\mathbb{A})} |\mathcal{W}^{U_{i-1},\psi}R(f)\Theta_\varphi(h)\mathcal{F}_\sigma(h)|\, dh\\
&= \int_{\mathbf{K}_H}\int_{(M_i)_H\backslash(M_i)_H(\mathbb{A})}\int_{Z_i\backslash T_i(\mathbb{A})}\\
&\int_{(U_i)_H\backslash(U_i)_H(\mathbb{A})} \delta_{(P_i)_H}(t)^{-1}|\mathcal{W}^{U_{i-1},\psi}R(f)\Theta_\varphi(utmk)\mathcal{F}_\sigma(tk)|\, du\, dt\, dm\, dk\\
&\leq \int_{Z_i\backslash T_i(\mathbb{A})} \delta_{(P_i)_H}(t)^{-1}\Phi_0(t_1/t_2,\dots,t_{i-1}/t_i)\sum_{\gamma\in F^*}\phi_0(\gamma {t_1}^{-1})\, dt.
\end{aligned}$$

To show that the last integral converges we make a change of variables $z_0 = t_1^{-1}$, $z_j = t_j/t_{j+1}$, $j = 1,\dots,i-1$. The integral becomes

$$\int_{\mathbb{I}_F^i} |z_0|^{\beta_0}\cdots|z_{i-1}|^{\beta_{i-1}}\Phi_0(z_1,\dots,z_{i-1})\phi_0(z_0)\, dz_0\, dz_1\cdots dz_{i-1}$$

where $\beta_0,\beta_1,\dots,\beta_{i-1} > 1$. The convergence follows. ∎

Remark 1: Formally, the same method would give

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\phi)\theta_{\Phi,\sigma}(h)\, dh = \int_{Z_M U_0^H(\mathbb{A})\backslash H(\mathbb{A})} \mathcal{W}^{U_0^M,\psi}M_{-1}\phi(h)\mathcal{F}_\sigma(h)\, dh.$$

However, there are some subtle convergence issues which is the reason we have to resort to pseudo-Eisenstein series (which are rapidly decreasing).

To continue the proof of Theorem 3 we write for simplicity φ instead of $R(f)\varphi$. Using (8) we write (15) as a sum of two terms

$$\int \mathcal{W}^{U_0^M,\psi}\varphi(h)\mathcal{F}_\sigma(h)\, dh, \tag{17}$$

$$\int \mathcal{W}^{U_0^M,\psi}M\varphi(h)\mathcal{F}_\sigma(h)\, dh, \tag{18}$$

provided that they both converge. We rewrite (17) using the Iwasawa decomposition as

$$\int_{\mathbf{K}_H}\int_{Z_M U_0^{M_H}(\mathbb{A})\backslash M_H(\mathbb{A})^{(1)}}\int_{\mathfrak{a}_M}\mathcal{W}^{U_0^M,\psi}\varphi(e^X mk)e^{-2\langle\rho_{P_H},X\rangle}\mathcal{F}_\sigma(e^X mk)\,dX\,dm\,dk.$$

The inner integral is

$$\int_{\mathfrak{a}_M} e^{-2\langle\rho_{P_H},X\rangle}\mathcal{W}^{U_0^M,\psi}\varphi(e^X mk)\int_{\mathbb{I}_F}\Phi(tv_0e^X mk)\mathcal{C}_\sigma(\log(|t|^n))\,d^*t\,dX.$$

Changing the variable of integration to $x=\langle\varpi_{2n}/2,X\rangle$ we get

$$\begin{aligned}&\int_{\mathbb{R}} e^{-(2n+2)x}\mathcal{W}^{U_0^M,\psi}\varphi(\alpha^\vee(e^{2x})mk)\int_{\mathbb{I}_F}\Phi(te^{-x/n}v_0mk)\mathcal{C}_\sigma(\log(|t|^n))\,d^*t\,dx\\ &=\int_{\mathbb{I}_F}\Phi(tv_0mk)\\ &\int_{\mathbb{R}}\delta_{P(A)}^{-1/2}(\alpha^\vee(e^{2x}))\mathcal{W}^{U_0^M,\psi}\varphi(\alpha^\vee(e^{2x})mk)e^{-x}\mathcal{C}_\sigma(\log(|t|^n)+x)\,dx\,d^*t.\end{aligned}$$

By Fourier inversion it can be written as

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathbb{I}_F}\Phi(tv_0mk)\\ &\qquad\int_{\mathbb{R}}\int_{\mathrm{Re}\,s=s_0}\mathcal{W}^{U_0^M,\psi}\hat\varphi_{-s}(mk)e^{-sx}e^{-x}\mathcal{C}_\sigma(\log(|t|^n)+x)\,dx\,ds\,dt\\ &=\int_{\mathbb{I}_F}\Phi(tv_0mk)\int_{\mathrm{Re}\,s=s_0}\mathcal{W}^{U_0^M,\psi}\hat\varphi_{-s}(mk)\sigma(s)|t|^{n(s+1)}\,ds\,dt\\ &=\int_{\mathrm{Re}\,s=s_0}\mathcal{W}^{U_0^M,\psi}\hat\varphi_{-s}(mk)\sigma(s)\phi_{\Phi,s}(mk)\,ds\end{aligned}$$

for any s_0. Altogether, we get

$$\int_{\mathrm{Re}\,s=s_0}\sigma(s)\int_{\mathbf{K}_H}\int_{Z_M U_0^{M_H}(\mathbb{A})\backslash M_H(\mathbb{A})^{(1)}}\mathcal{W}^{U_0^M,\psi}\hat\varphi_{-s}(mk)\phi_{\Phi,s}(mk)\,dm\,dk\,ds,$$

which by the main result of [BF90] is equal to

$$\int_{\mathrm{Re}\,s=s_0}\sigma(s)\int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^{(1)}}\mathbb{E}(mk,2s+1)\hat\varphi_{-s}(mk)\,dm\,dk\,ds$$

provided that $s_0>0$. Similarly, by (9) the term (18) is equal to

$$\int_{\mathrm{Re}\,s=s_0}\sigma(s)\int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^{(1)}}\mathbb{E}(mk,2s+1)M(s)\hat\varphi_s(mk)\,dm\,dk\,ds$$

provided that s_0 is sufficiently large. Shifting the contour of integration past $s = 1$ results in adding the contribution

$$2\pi i \cdot \sigma(1) \int_{\mathbf{K}_H} \int_{M_H \backslash M_H(\mathbb{A})^{(1)}} \mathbb{E}(mk, 3) M_{-1} \hat{\varphi}_1(mk) \; dm \; dk$$

of the residue. Altogether, we obtain the second formula of Theorem 3.

The steps following Lemma 2 are justified by the convergence (for $s_0 >> 0$) of

$$\int_{\mathrm{Re}\, s = s_0} \int_{\mathbf{K}_H} \int_{Z_M U_0^{M_H}(\mathbb{A}) \backslash M_H(\mathbb{A})^{(1)}} |\mathcal{W}^{U_0^M, \psi} \hat{\varphi}_{-s}(mk)| \phi_{\Phi, s_0}(mk) \; dm \; dk \; ds$$

(assuming, as we may, that $\Phi \geq 0$) and a similar term where $\hat{\varphi}_{-s}$ is replaced by $M(s)\hat{\varphi}_s$. Except for the integrations over s and k, which are harmless, this is implicit in [BF90]. In any case the argument is similar to the convergence part in the proof of Lemma 2.

7. Unitary groups

Let now $\mathbb{U} = F^{2n}$ be a symplectic vector space with respect to

$$\begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix}$$

and let E/F be a quadratic extension. We consider the space $\mathbb{V} = E^{2n}$ with the skew-Hermitian form obtained by extension of scalars. It defines a quasi-split unitary group $G = U(2n)$ (acting, as usual, on the right) which contains the period subgroup $H = Sp_n = G \cap GL_{2n}(F)$. Let P be maximal parabolic subgroup $P = MU$ stabilizing the maximal isotropic subspace

$$\mathbb{V}' = \{(0, \dots, 0, x_1, \dots, x_n) : x_i \in E\}.$$

Then $M \simeq GL_n(E)$ and $M_H = M \cap H \simeq GL_n(F)$. This is a twisted analogue of the case considered in [JLR03]. The Eisenstein series $\mathbb{E}_\Phi(\bullet, s)$ and $\mathcal{E}_\Phi(\bullet, s)$ are defined in the same way.

Let π be a cuspidal irreducible representation of $M(\mathbb{A})$. As before, $E_{-1}\varphi$ will denote the residue of the Eisenstein series induced from π (this time, at ϖ_n). We claim that Theorems 1, 2 and 3 remain true in this context with similar proofs. For example, to prove Theorem 3 we use once again the method of [AGR93]. We write

$$\begin{aligned} \mathfrak{P}_\Phi(\sigma, \varphi) &= \int_{Q_H \backslash H(\mathbb{A})} \Theta_\varphi(h) \mathcal{F}_{\Phi, \sigma}(h) \; dh \\ &= \int_{V_H(\mathbb{A}) L_H \backslash H(\mathbb{A})} \mathcal{F}_{\Phi, \sigma}(h) \int_{V_H \backslash V_H(\mathbb{A})} \Theta_\varphi(vh) \; dv \; dh \end{aligned}$$

and expand the inner integral along the Abelian group $V_H(\mathbb{A})V\backslash V(\mathbb{A})$. The contribution from the trivial character vanishes by the cuspidality of π. The other characters form a single orbit under L_H. We take the character $\chi(x_{i,j}) = \psi(x_{1,2} + x_{2n-1,2n})$ as a representative. Its stabilizer is

$$S_\chi = \{\begin{pmatrix} a & & & & \\ & a & * & * & \\ & & * & * & \\ & & & a^{-1} & \\ & & & & a^{-1} \end{pmatrix} : a \in F^*\} = L'V'.$$

Thus, we can write the above as

$$\int_{V_H(\mathbb{A})S_\chi\backslash H(\mathbb{A})} \mathcal{F}_{\Phi,\sigma}(h)E_{-1}^{\chi}(h,\varphi)\, dh = \int_{V_H(\mathbb{A})V'(\mathbb{A})L'\backslash H(\mathbb{A})} \mathcal{F}_{\Phi,\sigma}(h)\int_{V'\backslash V'(\mathbb{A})} E_{-1}^{\chi}(vh,\varphi)\, dv\, dh.$$

Again, we expand along the corresponding unipotent radical of G and use the cuspidality of π. Continuing this way we ultimately get

$$\mathfrak{P}_\Phi(\sigma,\varphi) = \int_{Z_M U_0^H(\mathbb{A})\backslash H(\mathbb{A})} \mathcal{W}^{U_0^M,\psi}\Theta_\varphi^U(h)\mathcal{F}_\sigma(h)\, dh.$$

The argument continues as before taking into account the identity

$$\int_{M_H\backslash M_H(\mathbb{A})^{(1)}} \phi(m)\mathbb{E}_\Phi(m,s)\, dm = \int_{Z_{M_H} U_0^{M_H}(\mathbb{A})\backslash M_H(\mathbb{A})^{(1)}} \mathcal{W}^{U_0^M,\psi}\phi(m)\phi_{\Phi,s'}^{\mathbb{V}}(m)\, dm,$$

valid for any cusp form ϕ on $M_H(\mathbb{A})$ ([Fli88]).

Theorem 1 follows exactly as in §5. The argument of [JLR03] now proves Theorem 2 provided we have the analogue of the additional identity (6). The latter is a part of work in progress by Tanai.

8. A Whittaker version

Similarly to [JLR03] we also have a Whittaker version of Theorem 2. In fact, for that we only need to assume the formula (6). So let us go back to the more general setup where $P = MU$ is a maximal parabolic of a quasi-split reductive group G over F. Let π be a cuspidal representation of $M(\mathbb{A})$ and $E(g,\varphi,\lambda)$ be the Eisenstein series induced from π. Suppose that π is generic with respect to a non-degenerate character ψ of $U_0 \cap M$. Let ϖ be the fundamental weight corresponding to P. The relation between intertwining operators and L-functions and the work of Shahidi on L-functions of generic representations suggest the following conjecture, which is probably known at least for classical groups.

CONJECTURE 1: *The only possible poles of $E(\cdot,\varphi,\lambda)$ for $\operatorname{Re}\lambda > 0$ are at $\lambda_0 = \frac{1}{2}\varpi$ or $\lambda_0 = \varpi$. At most one of them can occur. Moreover, the residue representation Π is irreducible.*

For a state-of-the-art discussion of this conjecture and related questions we refer the reader to a recent paper by Kim and Shahidi, and the reference therein ([KS03]). Note that if π is not generic then there may be poles farther away (presumably, at half-integer multiples of ϖ), although we do not know if more than one can occur.

In any case, let $E_{-1}(\cdot,\varphi)$ be the residue of $E(\cdot,\varphi,\lambda)$ at λ_0. Suppose that H is a subgroup of G such that $P_H = M_H U_H$ and that the relation (6) is satisfied. Extend ψ to a (degenerate) character of U_0, trivial on U. The argument of [JLR03, Theorem 4] is independent of Conjecture 1 and carries over verbatim. It yields the following

THEOREM 4: *We have*

$$\mathcal{B}^{\Pi}_{\ell_H,\overline{\mathcal{W}^{\psi,U}}}(f) = \mathcal{B}^{\pi}_{\ell_{M_H},\overline{\mathcal{W}^{\psi,U_0^M}}}(f') \tag{19}$$

where f' is the function on $M(\mathbb{A})$ defined by

$$f'(m) = e^{\langle\lambda_0+\rho_P,H_M(m)\rangle}\cdot\int_{\mathbf{K}_H}\int_{U(\mathbb{A})} f(k'mu)\,du\,dk'.$$

This is the spectral identity pertaining to the relative trace formula identity

$$\int_{H\backslash H(\mathbb{A})}\int_{U_0\backslash U_0(\mathbb{A})} K_f(h,u)\psi(u)\,du\,dh = \int_{M_H\backslash M_H(\mathbb{A})^{(1)}}\int_{U_0^M\backslash U_0^M(\mathbb{A})} K_{f'}(m,u')\psi(u')\,du\,dm.$$

Its geometric side was considered in [JR92a], [JMR99].

Apart from the cases considered above, the relation (6) holds in other situations. It is based on an analysis of the double cosets $P\backslash G/H$. Namely, one writes

$$\int_{H\backslash H(\mathbb{A})}\Theta_\phi(h)\,dh = \sum_{\eta\in P\backslash G/H}\int_{H\cap\eta^{-1}P\eta\backslash H(\mathbb{A})}\phi(\eta h)\,dh.$$

In the relevant cases the trivial coset gives the only non-zero contribution, which is

$$\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^{(1)}} \hat{\phi}_{\lambda_0}(mk)\, dm\, dk$$

where $\lambda_0 = 2\rho_{P_H} - \rho_P$. In contrast, the computation of Theorem 1 is based on the regularization of the constant function (rather than E_{-1}), and seems to be more restrictive.

In any case, in [Jia98] and [GJ01] the following were considered:

(1) $M = GL_2$ and M_H is the diagonal subgroup.
(2) $M = GL_2 \times SO(3)$ and $M_H = GL(2)$ imbedded diagonally.
(3) $M = \{(g_1, g_2, g_3) \in GL(2)^3 : \det(g_2) = \det(g_3)\}$ and $M_H = GL(2)$ imbedded diagonally.
(4) $M = GSp(6)$ and $M_H = \{(g_1, g_2, g_3) \in GL(2)^3 : \det(g_1) = \det(g_2) = \deg(g_3)\}$.

The corresponding pairs (G, H) are respectively

(1) (G_2, SL_3) (long roots). (M contains the short simple root.)
(2) (SO_7, G_2).
(3) $(GSpin(8), G_2)$.*
(4) $(F_4, Spin(8))$.

The numerator of the intertwining operator ([Lan71]), the location of the pole, and the residue are correspondingly

(1) $L(s,\pi)\zeta(2s)L(3s,\pi)$, $s_0 = \frac{1}{2}$, $\lambda_{-1}L(\frac{1}{2},\pi)L(\frac{3}{2},\pi)$.
(2) $L(s,\pi_1\otimes\pi_2)L(2s,\pi_1,sym^2)$, $s_0 = 1$, $\lambda_{-1}L(1,\pi_1,sym^2)L(2,\pi_1,sym^2)$.
(3) $L(s,\pi_1\otimes\pi_2\otimes\pi_3)L(2s,\omega_{\pi_1}\omega_{\pi_2}\omega_{\pi_3})$, $s_0 = \frac{1}{2}$, $\lambda_{-1}L(\frac{1}{2},\pi_1\otimes\pi_2\otimes\pi_3)$.
(4) $L(s,\pi,Spin)L(2s,\pi,St)$, $s_0 = 1$, $\lambda_{-1}L(1,\pi,St)L(2,\pi,St)$.

In the second case the analogue of Theorems 1 and 2 holds [GL]. In contrast, in the first and third cases there are also cuspidal presentations which are H-distinguished. (We do not know whether this is the situation in the fourth case as well.) Thus, in these cases Theorem 2 has to be reconsidered.

References

[AGR93] A. Ash, D. Ginzburg and S. Rallis, *Vanishing periods of cusp forms over modular symbols,* Mathematische Annalen **296** (1993), 709–723.

[Art80] J. Arthur, *A trace formula for reductive groups. II. Applications of a truncation operator,* Compositio Mathematica **40** (1980), 87–121.

* Strictly speaking $PGO(8)$ was considered instead, but everything carries over to $GSpin(8)$.

[BF90] D. Bump and S. Friedberg, *The exterior square automorphic L-functions on* GL(n), in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, Israel Mathematical Conference Proceedings, Vol. 3, Weizmann Science Press of Israel, Jerusalem, 1990, pp. 47–65.

[DM78] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bulletin des Sciences Mathématiques. 2e Série **102** (1978), 307–330.

[Fli88] Y. Z. Flicker, *Twisted tensors and Euler products*, Bulletin de la Société Mathématique de France **116** (1988), 295–313.

[GJ01] D. Ginzburg and D. Jiang, *Periods and liftings: From G_2 to C_3*, Israel Journal of Mathematics **123** (2001), 29–59.

[GL] D. Ginzburg and E. Lapid, *On a conjecture of Jacquet–Lai–Rallis; some exceptional cases*, preprint.

[GRS99] D. Ginzburg, S. Rallis and D. Soudry, *On explicit lifts of cusp forms from* GL_m *to classical groups*, Annals of Mathematics (2) **150** (1999), 807–866.

[HC68] Harish-Chandra, *Automorphic Forms on Semisimple Lie Groups*, Notes by J. G. M. Mars, Lecture Notes in Mathematics **62**, Springer-Verlag, Berlin, 1968.

[Jac] H. Jacquet, *A guide to the relative trace formula*, preprint.

[Jac97] H. Jacquet, *Automorphic spectrum of symmetric spaces*, in *Representation Theory and Automorphic Forms (Edinburgh, 1996)*, American Mathematical Society, Providence, RI, 1997, pp. 443–455.

[Jia98] D. Jiang, *G_2-periods and residual representations*, Journal für die reine und angewandte Mathematik **497** (1998), 17–46.

[JLR93] H. Jacquet, K. F. Lai and S. Rallis, *A trace formula for symmetric spaces*, Duke Mathematical Journal **70** (1993), 305–372.

[JLR03] H. Jacquet, E. Lapid and S. Rallis, *A spectral identity for skew symmetric matrices*, in *Contributions to Automorphic Forms, Geometry and Number Theory*, Shalikafest 2002, A Supplemental Volume to the American Journal of Mathematics, 2003.

[JMR99] D. Jiang, Z. Mao and S. Rallis, *A relative Kuznietsov trace formula on G_2*, Manuscripta Mathematica **99** (1999), 411–423.

[JR92a] H. Jacquet and S. Rallis, *Kloosterman integrals for skew symmetric matrices*, Pacific Journal of Mathematics **154** (1992), 265–283.

[JR92b] H. Jacquet and S. Rallis, *Symplectic periods*, Journal für die reine und angewandte Mathematik **423** (1992), 175–197.

[JS81] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic forms. II*, American Journal of Mathematics **103** (1981), 777–815.

[JS90] H. Jacquet and J. Shalika, *Exterior square L-functions*, in *Automorphic Forms, Shimura Varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988)*, Academic Press, Boston, MA, 1990, pp. 143–226.

[KS03] H. H. Kim and F. Shahidi, *On the holomorphy of certain L-funtions*, in *Contributions to Automorphic Forms, Geometry and Number Theory*, Shalikafest 2002, A Supplemental Volume to the American Journal of Mathematics, 2003.

[Lan71] R. P. Langlands, *Euler Products*, A James K. Whittemore Lecture in Mathematics given at Yale University, 1967, Yale Mathematical Monographs, 1, Yale University Press, New Haven, Conn., 1971.

[Lap] E. Lapid, *On the spectral expansion of Jacquet's relative trace formula*, preprint.

[LR03] E. Lapid and J. Rogawski, *Periods of Eisenstein series: The Galois case*, Duke Mathematical Journal **120** (2003), 153–226.

[MW95] C. Mœglin and J.-L. Waldspurger, *Spectral Decomposition and Eisenstein Series*, Une paraphrase de l'Écriture [A paraphrase of Scripture], Cambridge University Press, Cambridge, 1995.